

# A MATRIX RING WITH COMMUTING GRAPH OF MAXIMAL DIAMETER

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**ABSTRACT.** The commuting graph of a semigroup is the set of non-central elements; the edges are defined as pairs  $(u, v)$  satisfying  $uv = vu$ . We provide an example of a field  $\mathcal{F}$  and an integer  $n$  such that the commuting graph of  $\text{Mat}_n(\mathcal{F})$  has maximal possible diameter, equal to six.

## 1. INTRODUCTION

The study of commuting graphs of different algebraic structures has attracted considerable attention in recent publications. There are a number of interesting results connecting this area of research with functional analysis [5], ring theory [3], semigroup theory [6], and some other branches of mathematics. A significant number of publications are devoted to commuting graphs arising from group theory [1] and linear algebra [4]. One of the interesting recent results was obtained in [16], where the following group-theoretical problem has been solved: What is the maximal possible diameter of the commuting graph of a finite group? Our paper deals with the linear algebraic version of this question. Let  $\mathbb{F}$  be a field and  $\text{Mat}_n(\mathbb{F})$  be the algebra of  $n \times n$  matrices over  $\mathbb{F}$ . We denote the commuting graph of  $\text{Mat}_n(\mathbb{F})$  by  $\Gamma(\mathbb{F}, n)$ . That is, the vertices of  $\Gamma(\mathbb{F}, n)$  are non-scalar matrices, and the edges are defined as pairs  $(U, V)$  satisfying  $UV = VU$ . S. Akbari, A. Mohammadian, H. Radjavi, and P. Raja proved in 2006 that the distances in commuting graphs of matrix algebras cannot exceed six, and they proposed the following conjecture.

**Conjecture 1.1.** [4, Conjecture 5] *If  $\Gamma(\mathbb{F}, n)$  is a connected graph, then its diameter does not exceed five.*

This conjecture is known to be true in the following cases:  $n = 4$  or  $n$  is prime [11, 2];  $\mathbb{F}$  is algebraically closed [4] or real closed [20];  $\mathbb{F}$  is finite and  $n$  is not a square of a prime [11]. The set of matrix pairs realizing the maximal distance is well understood in the case of algebraically closed field [10]. It was proved in [9] that  $\Gamma(\mathbb{F}_2, 9)$  is connected and has diameter at least five, showing that Conjecture 1.1 cannot be strengthened. Conjecture 1.1 has also been extensively investigated in the case when  $\mathbb{F}$  is a more general structure than a field. This conjecture is known to be true when  $\mathbb{F}$  is equal to  $\mathbb{Z}/m$  with non-prime  $m$  or taken from a sufficiently general class of semirings which includes the *tropical semiring* [12, 13, 15].

Conjecture 1.1 has remained open until now, and our paper aims to provide a counterexample for this conjecture. In Section 2, we construct a field over which Conjecture 1.1 fails. In Section 3, we construct a pair of matrices which realize the

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maximal distance, equal to six, in the corresponding commuting graph. In Section 4, we conclude the paper by proving a technical claim used in our argument.

## 2. THE FIELD

Let us recall some notation. An integral domain  $R$  is called a *gcd domain* if any two non-zero elements have a greatest common divisor. An element  $v$  in some extension of  $R$  is called *integral over  $R$*  if  $v$  is a root of a monic polynomial with coefficients in  $R$ . A ring  $R$  is called *local* if the set  $J = J(R)$  of all non-units is an ideal of  $R$ . A local ring  $R$  is *Henselian* if it satisfies the condition of *Hensel's lemma*. That is, for every monic polynomial  $f \in R[t]$  whose image  $\bar{f}$  in  $R[t]/J[t]$  factors into a product  $\bar{g}_1\bar{g}_2$  in which  $\bar{g}_1$  and  $\bar{g}_2$  are both monic and relatively prime, there exist monic polynomials  $g_1, g_2 \in R[t]$  which are relatively prime and satisfy  $f = g_1g_2$  and  $\bar{g}_i = g_i + J[t]$ .

Let us fix a prime number  $p \geq 73$ . We work with the power series ring  $\mathcal{R} = \mathbb{F}_3[[x_{11}, x_{12}, \dots, x_{3p-3p}, y, z]]$ , which is a gcd domain [8, Corollary 3.3] and also a local Henselian ring [23]. The ideal  $J(\mathcal{R})$  consists of series with zero constant term, so we have a natural isomorphism between  $\mathbb{F}_3$  and  $\mathcal{R}/J(\mathcal{R})$ . We denote by  $\mathcal{Q}$  the field of fractions  $\text{Quot } \mathcal{R}$  and by  $\mathcal{H}$  the algebraic closure of  $\mathcal{Q}$ . (We do not provide an explicit construction of  $\mathcal{H}$ , but we refer the reader to [22] for related questions.) We consider a polynomial  $\tilde{\varphi} = t^p + a_{p-1}t^{p-1} + \dots + a_1t + a_0$  which is irreducible over  $\mathbb{F}_3$ , and we define

$$\varphi = t^p + a_{p-1}t^{p-1} + \dots + a_2t^2 + (a_1 + y)t + (a_0 + z) \in \mathcal{R}[t],$$

where  $t$  is an indeterminate, and coefficients lie in  $\mathcal{R}$ . The polynomial  $\bar{\varphi}$  equals  $\tilde{\varphi}$  up to the above mentioned isomorphism between  $\mathbb{F}_3$  and  $\mathcal{R}/J(\mathcal{R})$ , so  $\bar{\varphi}$  is irreducible over  $\mathcal{R}/J(\mathcal{R})$ . Therefore,  $\varphi$  is irreducible over  $\mathcal{R}$  and, since Gauss' lemma holds in gcd domains [21, IV.4, Theorem 4.6],  $\varphi$  is irreducible over  $\mathcal{Q}$  as well.

**Lemma 2.1.** *Let  $\Phi$  be the set of all subfields  $F \subset \mathcal{H}$  such that (1)  $F$  is a separable extension of  $\mathcal{Q}$  and (2)  $\varphi$  is irreducible over  $F$ . Then  $\Phi$  has a maximal field  $\mathcal{F}$ .*

*Proof.* Let  $\mathcal{F}' = \{\mathcal{F}_i\}$  be a totally ordered subset of  $\Phi$ . Every element of  $\mathcal{F}_0 = \bigcup \mathcal{F}'$  is separable over  $\mathcal{Q}$ , so  $\mathcal{F}_0$  is a separable extension. Assuming  $\varphi = \varphi_1\varphi_2$  over  $\mathcal{F}_0$ , we enumerate the coefficients involved in  $\varphi_1$  and  $\varphi_2$  as  $c_1, \dots, c_k$ , and every  $c_j$  lies in some field  $F_j \in \mathcal{F}'$ . Since  $\varphi$  is irreducible over  $F_1 \cup \dots \cup F_k \in \mathcal{F}'$ , we see that either  $\varphi_1$  or  $\varphi_2$  is constant, which means that  $\varphi$  is irreducible over  $\mathcal{F}_0$ . Application of Zorn's lemma completes the proof.  $\square$

In what follows,  $\mathcal{F}$  denotes the field constructed in the previous lemma.

**Lemma 2.2.** *Any non-trivial separable extension of  $\mathcal{F}$  contains a root of  $\varphi$ .*

*Proof.* Let  $F$  be such an extension. Since the field  $\mathcal{H}$  is algebraically closed, we can assume without loss of generality that  $F \subset \mathcal{H}$ . By Lemma 2.1, there are non-constant monic polynomials  $\varphi_1, \varphi_2 \in F[t]$  such that  $\varphi = \varphi_1\varphi_2$ . We enumerate by  $c_1, \dots, c_k$  the coefficients appearing in  $\varphi_1$  and  $\varphi_2$ ; note that  $c_1, \dots, c_k$  are integral over  $\mathcal{R}$  because they are sums of products of roots of  $\varphi$ .

The ring  $R' = \mathcal{R}[c_1, \dots, c_k]$  is local [17, II.7, Ex. 7.7] and Henselian [7, III.4, Ex. 4c]. Since  $\varphi$  is reducible over  $R'$ , the polynomial  $\bar{\varphi}$  is reducible over  $R'/J(R')$ . Since finite extensions of finite fields are cyclic, the splitting field of  $\bar{\varphi}$  has degree  $p$  over  $\mathcal{R}/J(\mathcal{R}) \cong \mathbb{F}_3$ . So we see that  $\bar{\varphi}$  splits (and, in particular, has a root

$u)$  over  $R'/J(R')$ . By Hensel's lemma, there exists  $\rho \in R'$  such that  $\bar{\rho} = u$  and  $\varphi(\rho) = 0$ .  $\square$

Recall that, for every algebraic field extension  $K \supset L$ , the set of all elements  $l \in K$  that are separable over  $L$  forms the field  $K_{sep}$ , which is the unique separable extension of  $L$  over which  $K$  is purely inseparable [18]. The degree of the extension  $K \supset K_{sep}$  is called the *inseparable degree* of  $K \supset L$  and is denoted by  $[K : L]_i$ . The following theorem is an essential step in the proof of our main result.

**Theorem 2.3.** *The degree of any finite extension  $E \supset \mathcal{F}$  is either a power of 3 or a multiple of  $p$ . If the degree of  $E$  is a multiple of  $p$ , then  $E$  contains a subfield  $E'$  which has degree  $p$  over  $\mathcal{F}$ .*

*Proof.* The degree of any purely inseparable extension is a power of the characteristic, so we can assume that  $E_{sep} \neq \mathcal{F}$ . By Lemma 2.2, the polynomial  $\varphi$  has a root  $\xi \in E_{sep}$ , and then  $E' = \mathcal{F}(\xi)$  has degree  $p$  over  $\mathcal{F}$ .  $\square$

By Theorem 6 of [2], the second assertion of Theorem 2.3 is a sufficient condition for the graph  $\Gamma(\mathcal{F}, 3p)$  to be connected. Let us recall that the equality  $[K_1 : K_3]_i = [K_1 : K_2]_i [K_2 : K_3]_i$  holds for any tower  $K_1 \supset K_2 \supset K_3$  of algebraic extensions [18]. We denote by  $\psi(\tau)$  the polynomial  $\varphi(\tau^3)$  and by  $\theta$  a root of  $\psi$ .

*Remark 2.4.* Since every root of  $\psi$  has multiplicity three, the only monic separable polynomial of degree at least  $p$  that divides  $\psi$  is  $\sqrt[3]{\psi}$ . Note that the elements  $\sqrt[3]{y}$  and  $\sqrt[3]{z}$  belong to any field containing the coefficients of  $\sqrt[3]{\psi}$ .

**Lemma 2.5.** *The polynomial  $\psi$  is irreducible over  $\mathcal{F}$ .*

*Proof.* Assume that  $\psi$  is divisible by a polynomial  $\psi_1 \in \mathcal{F}[\tau]$  of degree  $d \notin \{0, 3p\}$ . If  $\psi_1 = \varphi_1(\tau^3)$  for some  $\varphi_1 \in \mathcal{F}[t]$ , then  $\varphi$  is divisible by  $\varphi_1$ , a contradiction. In the rest of our proof, we assume without loss of generality that  $\psi_1$  is irreducible; if  $\psi_1$  is inseparable, then it satisfies the assumption of the previous sentence. If  $\psi_1$  is separable and  $d < p$ , then the polynomial  $\psi_2 = \psi_1^3$  in turn satisfies the assumption of the second sentence.

Finally, if  $\psi_1$  is separable and  $d \geq p$ , then  $\sqrt[3]{y} \in \mathcal{F}$  by Remark 2.4. The definition of  $\mathcal{F}$  implies that  $\sqrt[3]{y}$  is separable over  $\mathcal{Q}$ , and this is a contradiction.  $\square$

**Lemma 2.6.** *Let  $\theta$  be a root of  $\psi$  and  $K$  a field. If  $\mathcal{F} \subsetneq K \subsetneq \mathcal{F}(\theta)$ , then  $K = \mathcal{F}(\theta^3)$ .*

*Proof.* Since  $\psi$  is irreducible, we get  $[\mathcal{F}(\theta) : \mathcal{F}] = 3p$ . If  $[K : \mathcal{F}] = p$ , then  $K$  is the field of all separable elements of  $\mathcal{F}(\theta)$ . In this case,  $K = \mathcal{F}(\theta^3)$ .

If  $[K : \mathcal{F}] = 3$ , then  $[\mathcal{F}(\theta) : K] = p$ . In this case,  $\theta$  is a root of an irreducible polynomial  $\chi \in K[t]$  of degree  $p$ , and  $\chi$  is separable. By Remark 2.4, we have  $\chi = \sqrt[3]{\psi}$  which implies  $\sqrt[3]{y}, \sqrt[3]{z} \in K$ . Note that the extension  $\mathcal{Q}(\sqrt[3]{y}, \sqrt[3]{z}) \supset \mathcal{Q}$  has inseparability degree nine, which means that  $[K : \mathcal{F}]_i \geq 9$  because  $\mathcal{F} \supset \mathcal{Q}$  is a separable extension. Therefore,  $[K : \mathcal{F}] \neq 3$ .  $\square$

### 3. THE MATRICES

Let us recall a standard result we need in the rest of our paper. We denote by  $\mathcal{C}(A)$  the *centralizer* of a matrix  $A \in \text{Mat}_n(\mathbb{F})$ , that is, the set of all matrices that commute with  $A$ . Clearly,  $\mathcal{C}(A)$  is an  $\mathbb{F}$ -linear subspace of  $\text{Mat}_n(\mathbb{F})$ , so we can speak of the dimension of  $\mathcal{C}(A)$ . A standard result of matrix theory states that there exists a nonsingular matrix  $Q \in \text{Mat}_n(\mathbb{F})$  such that  $Q^{-1}AQ$  has *rational normal form*. In

other words, we have  $Q^{-1}AQ = \text{diag}[L(f_1), L(f_2), \dots, L(f_k)]$ , where  $L(f)$  denotes the companion matrix of a polynomial  $f$ , and the polynomials  $f_i$  satisfy  $f_{i+1}|f_i$  for all  $i$ . These polynomials  $f_i$  are called the *invariant factors* of  $A$ .

**Theorem 3.1.** [14, VIII.2, Theorem 2] *The centralizer of a matrix  $A \in \text{Mat}_n(\mathbb{F})$  has dimension  $n_1 + 3n_2 + \dots + (2k-1)n_k$ , where  $n_1 \geq \dots \geq n_k$  are the degrees of the invariant factors of  $A$ .*

**Corollary 3.2.** [14, VIII.2, Corollary 1] *If the characteristic polynomial of  $A \in \text{Mat}_n(\mathbb{F})$  is irreducible, then  $\mathcal{C}(A) = \mathbb{F}[A]$ .*

**Claim 3.3.** *Let  $\mathcal{G} = \text{diag}(G, G, G)$  be the block-diagonal matrix, where  $G$  denotes the companion matrix of  $\varphi$ . If  $X \in \text{Mat}_{3p}(\mathcal{F})$  is the matrix whose  $(i, j)$ th entry equals  $x_{ij}$ , then the distance between  $\mathcal{G}$  and  $X^{-1}\mathcal{G}X$  in  $\Gamma(\mathcal{F}, 3p)$  is at least four.*

Before we start proving Claim 3.3, we deduce the main result from it.

**Theorem 3.4.** *Let  $\Psi$  be an  $\mathcal{F}$ -linear operator on  $\mathcal{F}^{3p}$  with characteristic polynomial  $\psi$ , and denote by  $\Phi \in \text{Mat}_{3p}(\mathcal{F})$  the matrix of  $\Psi$  written with respect to a basis such that  $\Psi^3$  has rational normal form. Then the distance between  $\Phi$  and  $X^{-1}\Phi X$  in  $\Gamma(\mathcal{F}, 3p)$  is at least six.*

*Proof.* Suppose  $\Phi \leftrightarrow M_1 \leftrightarrow M_2 \leftrightarrow M_3 \leftrightarrow M_4 \leftrightarrow X^{-1}\Phi X$  is a path in  $\Gamma(\mathcal{F}, 3p)$ . Corollary 3.2 implies  $\mathcal{C}(\Phi) = \mathcal{F}[\Phi]$ ; since  $\psi$  is irreducible,  $\mathcal{F}[\Phi]$  is isomorphic to  $\mathcal{F}(\theta)$  as a field. By Lemma 2.6, the only matrices in  $\mathcal{F}[\Phi]$  whose centralizer is larger are those in  $\mathcal{F}[\Phi^3]$ ; moreover, all non-scalar matrices in  $\mathcal{F}[\Phi^3]$  are polynomials in each other which means that these matrices have the same centralizer. Therefore, we can assume without loss of generality that  $M_1 = \Phi^3$  and  $M_4 = X^{-1}\Phi^3 X$ . The minimal polynomial of  $\Phi^3$  is  $\varphi$ , so  $M_1$  is the block-diagonal matrix with three blocks equal to the companion matrix of  $\varphi$ . The rest follows from Claim 3.3.  $\square$

#### 4. PROOF OF CLAIM 3.3

This is a final section which is intended to prove Claim 3.3. Let  $A$  be a matrix over some extension of a field  $\mathbb{F}$ ; we denote by  $\text{trdeg}(A, \mathbb{F})$  the transcendence degree of the field extension obtained from  $\mathbb{F}$  by adjoining the entries of  $A$ . In the special case when  $A$  is a matrix over  $\mathcal{F}$ , we write simply  $\text{trdeg}(A)$  instead of  $\text{trdeg}(A, \mathbb{F}_3(y, z))$ . In particular, we have  $\text{trdeg}(X) = 9p^2$  and  $\text{trdeg}(\mathcal{G}) = 0$  for the matrices defined in Claim 3.3.

**Lemma 4.1.** *Let  $\mathbb{F}'$  be an extension of  $\mathbb{F}$ . If matrices  $A, B, C$  over  $\mathbb{F}'$  satisfy  $B = C^{-1}AC$ , then  $\text{trdeg}(C, \mathbb{F}) \leq \text{trdeg}(A, \mathbb{F}) + \text{trdeg}(B, \mathbb{F}) + \dim \mathcal{C}(A)$ .*

*Proof.* Let us denote by  $K$  the algebraic closure of the field generated by entries of  $A$  and  $B$ . Applying the Jordan normal form theorem, we see that there is a  $J \in \text{Mat}_n(K)$  such that  $J^{-1}BJ = A$ . Since  $CJ \in \mathcal{C}(A)$ , the entries of  $CJ$  can be written as  $K$ -linear functions of  $c = \dim \mathcal{C}(A)$  elements of  $\mathbb{F}'$ . Thus, the entries of  $C$  are algebraic in these  $c$  elements and the entries of  $J$ . The entries of  $J$  are in turn algebraic in the entries of  $A$  and  $B$ , so we have  $\text{trdeg}(C, \mathbb{F}) \leq \text{trdeg}(A, \mathbb{F}) + \text{trdeg}(B, \mathbb{F}) + c$ .  $\square$

**Corollary 4.2.** *Assume  $\mathbb{F} \subset \mathbb{F}'$  and matrices  $A, C, D$  over  $\mathbb{F}'$  satisfy  $DC^{-1}AC = C^{-1}ACD$ . Then  $\text{trdeg}(C, \mathbb{F}) \leq \text{trdeg}(A, \mathbb{F}) + \text{trdeg}(D, \mathbb{F}) + \dim \mathcal{C}(A) + \dim \mathcal{C}(D)$ .*

*Proof.* Let  $K$  be the field obtained from  $\mathbb{F}$  by adjoining the entries of  $D$ . Since the matrix  $B = C^{-1}AC$  belongs to  $\mathcal{C}(D)$ , the entries of  $B$  can be written as  $K$ -linear functions of  $c = \dim \mathcal{C}(D)$  elements of  $\mathbb{F}'$ . We get  $\text{trdeg}(B, \mathbb{F}) \leq \text{trdeg}(D, \mathbb{F}) + \dim \mathcal{C}(D)$ , and now it suffices to substitute this into the right-hand side of the inequality of Lemma 4.1.  $\square$

**Lemma 4.3.** *Every matrix  $A \in \text{Mat}_n(\mathcal{F})$  of rank  $r$  satisfies  $\text{trdeg}(A) \leq 2nr - r^2$ .*

*Proof.* We have  $A = BC$ , where  $B \in \mathcal{F}^{n \times r}$  and  $C \in \mathcal{F}^{r \times n}$ . Since the product  $BC$  is preserved by the transformation  $(B, C) \rightarrow (BD, D^{-1}C)$ , we can assume that  $C$  has a unit  $r \times r$  submatrix.  $\square$

**Lemma 4.4.** *Assume that a matrix  $N$  over  $\mathcal{F}$  has one of the forms*

$$N_1 = \begin{pmatrix} F_1 & 0 & 0 \\ 0 & F_2 & F_3 \\ 0 & F_4 & F_5 \end{pmatrix}, \quad N_2 = \begin{pmatrix} F_6 & F_8 & F_0 \\ 0 & F_7 & F_9 \\ 0 & 0 & F_6 \end{pmatrix},$$

where  $F_j \in \text{Mat}_p(\mathcal{F})$ . If  $\text{rank}(N) = p$ , then  $\text{trdeg}(N) \leq 3.75p^2$ .

*Proof.* 1. If  $F_1$  has rank  $k$ , then the bottom right  $2p \times 2p$  submatrix of  $N_1$  has rank  $p - k$ ; by Lemma 4.3, we get  $\text{trdeg}(N_1) \leq 2pk - k^2 + 4p(p - k) - (p - k)^2 = 3p^2 - 2k^2 \leq 3p^2$ .

2. We have  $\text{rank } F_6 \leq p/2$ , and Lemma 4.3 shows that  $\text{trdeg}(F_6) \leq 0.75p^2$ . Finally, the upper right  $2p \times 2p$  submatrix of  $N_2$  has rank at most  $p$ , so its transcendence degree does not exceed  $3p^2$  again by Lemma 4.3.  $\square$

**Lemma 4.5.** *Let  $\mathbb{F}$  be a field over which polynomials of degrees  $1, \dots, d - 1$  have roots. If a matrix  $M \in \text{Mat}_n(\mathbb{F})$  has no eigenvalue in  $\mathbb{F}$ , then  $\dim \mathcal{C}(M) \leq n^2/d$ .*

*Proof.* Let  $n_1 \geq \dots \geq n_k$  be the degrees of the invariant factors. Since  $M$  has no eigenvalue in  $\mathbb{F}$ , we have  $n_i \geq d$  for all  $i \in \{1, \dots, k\}$ . Denoting  $\delta_i = n_i/d$ , we get  $\delta_i \geq 1$  and also  $\delta_1 + \dots + \delta_k = n/d$ . Now we have  $2i + 1 \leq 2\delta_1 + \dots + 2\delta_i + 1$  for all  $i$ , and Theorem 3.1 implies  $\dim \mathcal{C}(M)/d = \delta_1 + 3\delta_2 + \dots + (2k - 1)\delta_k$ , which is  $\leq \delta_1 + (2\delta_1 + 1)\delta_2 + (2\delta_1 + 2\delta_2 + 1)\delta_3 + \dots + (2\delta_1 + \dots + 2\delta_{k-1} + 1)\delta_k = \delta_1 + \dots + \delta_k + 2 \sum_{i < j} \delta_i \delta_j \leq (\delta_1 + \dots + \delta_k)^2 = n^2/d^2$ .  $\square$

**Lemma 4.6.** *Let  $N \in \mathcal{C}(\mathcal{G})$  be a non-scalar matrix, where  $\mathcal{G}$  is the matrix from Claim 3.3. If  $\dim \mathcal{C}(N) > 3p^2$ , then*

- (1) *there is a non-scalar matrix  $M \in \mathcal{C}(\mathcal{G})$  such that  $\mathcal{C}(N) \subset \mathcal{C}(M)$  and either  $M^2 = M$  or  $M^2 = 0$ ;*
- (2) *there is a matrix  $U \in \mathcal{C}(\mathcal{G})$  such that  $\mathcal{C}(UNU^{-1})$  is contained in the centralizer of one of the matrices*

$$M_1 = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where every block has size  $p \times p$ .

*Proof.* One can note that the elements of  $\mathcal{C}(\mathcal{G})$  are the  $3p \times 3p$  matrices such that, when they are written as  $3 \times 3$  arrays of  $p \times p$  blocks, all of those  $p \times p$  blocks commute with  $G$ . By Corollary 3.2, these blocks are actually elements of  $\mathcal{F}[G]$ ; in other words, we can think of  $\mathcal{C}(\mathcal{G})$  as the set of  $3 \times 3$  matrices over the field  $\mathcal{F}[G]$ .

If the assertion (1) of the lemma is true, we can compute the Jordan normal form of  $M$  as a  $3 \times 3$  matrix over  $\mathcal{G}$ . If  $M^2 = 0$ , then the Jordan normal form of  $M$  is the  $M_2$  of statement (2); if  $M^2 = M$ , then  $\mathcal{C}(M) = \mathcal{C}(I - M)$  and either  $M$  or  $I - M$  has Jordan normal form equal to  $M_1$ . Now it suffices to prove (1).

If  $N$  has no eigenvalue in  $\mathcal{F}$ , then Lemma 4.5 with  $d = 3$  implies  $\dim \mathcal{C}(N) \leq 3p^2$ , which is a contradiction. (Lemma 4.5 is applicable because every degree two polynomial over  $\mathcal{F}$  is reducible by Theorem 2.3.) We see that  $N$  has an eigenvalue  $\lambda \in \mathcal{F}$ , and then  $L = N - \lambda I$  is non-invertible and  $\mathcal{C}(N) = \mathcal{C}(L)$ . If  $L$  is nilpotent, then either  $L$  or  $L^2$  is a nonzero square-zero matrix, and we have  $\mathcal{C}(L) \subset \mathcal{C}(L^2)$ .

If  $L$  is not nilpotent, then we construct the rational normal form of  $L^2$ . That is, we find a matrix  $U \in \mathcal{C}(\mathcal{G})$  such that  $UL^2U^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & L' \end{pmatrix}$ , where  $L'$  is either a  $p \times p$  or  $2p \times 2p$  invertible matrix. The matrix  $L_0$  obtained by replacing  $L'$  with the unit matrix of the relevant size is a polynomial in  $UL^2U^{-1}$ , so we get  $L_0 = L_0^2$  and  $\mathcal{C}(ULU^{-1}) \subset \mathcal{C}(UL^2U^{-1}) \subset \mathcal{C}(L_0)$ .  $\square$

**Lemma 4.7.** *For any  $U \in \mathcal{C}(\mathcal{G})$ , we have  $\text{trdeg}(U) \leq 9p$ .*

*Proof.* By the Cayley–Hamilton theorem, the matrix  $G^p$  (where  $G$  is as in Claim 3.3) is an  $\mathcal{F}$ -linear combination of matrices  $I, G, \dots, G^{p-1}$ . In other words, any matrix  $P \in \mathcal{F}[G]$  can be written as  $\lambda_0 I + \lambda_1 G + \dots + \lambda_{p-1} G^{p-1}$  with  $\lambda_i \in \mathcal{F}$ . Since  $G$  is a companion matrix of a polynomial with coefficients in  $\mathbb{F}_3[x, y]$ , we get  $\text{trdeg}(G) = 0$  and  $\text{trdeg}(P) \leq p$ . As explained in the proof of Lemma 4.6, matrices in  $\mathcal{C}(\mathcal{G})$  are  $3p \times 3p$  matrices such that, when they are written as  $3 \times 3$  arrays of  $p \times p$  blocks, all of those nine blocks belong to  $\mathcal{F}[G]$ .  $\square$

We are now ready to complete the proof of Claim 3.3.

*Proof of Claim 3.3.* Suppose  $\mathcal{G} \leftrightarrow S \leftrightarrow X^{-1}TX \leftrightarrow X^{-1}\mathcal{G}X$  is a path in  $\Gamma(\mathcal{F}, 3p)$ . By Corollary 4.2 with  $X, T, S$  in the roles of  $C, A, D$ , respectively, we have  $\text{trdeg}(X) \leq \text{trdeg}(T) + \text{trdeg}(S) + \dim \mathcal{C}(T) + \dim \mathcal{C}(S)$ . Lemma 4.7 implies  $\text{trdeg}(T) + \text{trdeg}(S) \leq 18p$ , so that  $\dim \mathcal{C}(T) + \dim \mathcal{C}(S) \geq 9p^2 - 18p > 8p^2$  (since  $p > 18$ ).

Thus, at least one of  $\dim \mathcal{C}(S), \dim \mathcal{C}(T)$  must be  $> 3p^2$ . Suppose  $\dim \mathcal{C}(S) > 3p^2$ . Note that matrices commuting with  $M_1, M_2$  as in Lemma 4.6 have the forms  $N_1, N_2$  as in Lemma 4.4, respectively. Therefore, Lemma 4.6 implies  $\dim \mathcal{C}(S) \leq 5p^2$ , which implies  $\dim \mathcal{C}(T) > 3p^2$  by applying the conclusion of the above paragraph. If, instead of  $\dim \mathcal{C}(S) > 3p^2$ , we assume  $\dim \mathcal{C}(T) > 3p^2$ , then we similarly get  $\dim \mathcal{C}(S) > 3p^2$ . So both these bounds hold.

Since we can replace  $S$  and  $T$  by any non-scalar matrices with the same or larger centralizers, Lemma 4.6 allows us to assume that  $S = UM_iU^{-1}$  and  $T = V^{-1}M_jV$ , for some  $i, j \in \{1, 2\}$  and  $U, V \in \mathcal{C}(\mathcal{G})$ . Since any matrix commuting with  $M_i$  has the form  $N_i$  as in Lemma 4.4, we get  $X^{-1}TX = UN_iU^{-1}$ , which implies  $N_i = (VXU)^{-1}M_j(VXU)$ ; we have

$$(4.1) \quad \text{trdeg}(VXU) \leq \text{trdeg}(N_i) + \text{trdeg}(M_j) + \dim \mathcal{C}(M_j)$$

by Lemma 4.1. Since  $\text{rank } N_i = \text{rank } T = \text{rank } M_j = p$ , Lemma 4.4 implies  $\text{trdeg}(N_i) \leq 3.75p^2$ ; we also get  $\text{trdeg}(M_j) = 0$ ,  $\dim \mathcal{C}(M_j) = 5p^2$  straightforwardly. Finally, we note that  $\text{trdeg}(X) \leq \text{trdeg}(VXU) + \text{trdeg}(V^{-1}) + \text{trdeg}(U^{-1})$ , and Lemma 4.7 implies  $\text{trdeg}(V^{-1}) + \text{trdeg}(U^{-1}) \leq 18p$ . We get  $\text{trdeg}(VXU) \geq 9p^2 - 18p$ , so that (4.1) implies  $9p^2 - 18p \leq 3.75p^2 + 0 + 5p^2$ , i.e.,  $0.25p^2 \leq 18p$ . Since  $p \geq 73$ , this is a contradiction.  $\square$

The proof of the main result is now complete. Theorems 2.3 and 3.4 show that  $\Gamma(\mathcal{F}, 3p)$  is a connected graph with diameter greater than five, which disproves Conjecture 1.1. The distances in commuting graphs of matrix algebras cannot exceed six as Theorem 17 of [4] shows, so the diameter of  $\Gamma(\mathcal{F}, 3p)$  equals six.

The idea of our construction comes from the paper [19], which contains the proof of a statement similar to Lemma 2.6. I would like to thank the anonymous reviewers for their interest to this project, careful reading of the preliminary versions, and numerous helpful suggestions. I am grateful to Alexander Guterman for interesting discussions on this topic.

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